Russell: Paradox and Type Theory

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FRW Lecture 4. Friday wk. 4, HT14

1 The Paradoxes

The theory of types

‘Mathematical Logic as Based on the Theory of Types’ presents:¹

- A logical system—ramified type theory—developing the logic set out in Frege’s Grundgesetze.
- Applications of this logical system to mathematics.

Two reasons to type.

The following theory of symbolic logic recommended itself to me in the first instance by its ability to solve certain contradictions... But the theory in question seems not wholly dependent on this indirect recommendation...it has also...a certain consonance with common sense which makes it inherently credible. (p. 59)

Reason 1: Type theory avoids the inconsistency Frege fell into.

Reason 2: Type theory enjoys “consonance with common sense”.

Let’s consider just the first two of the seven contradictions Russell looks at:

(1) Epimenides’ paradox (the Liar)

The simplest form of the contradiction is afforded by the man who says ‘I am lying’; if he is lying, he is speaking the truth, and vice versa (ibid.)

(2) Russell’s paradox

Let $w$ be the class of all those classes which are not members of themselves. Then, whatever class $x$ may be ‘$x$ is a $w$’ is equivalent to ‘$x$ is not an $x$’.

Hence, giving $x$ the value $w$, ‘$w$ is a $w$’ is equivalent to ‘$w$ is not a $w$’. (ibid.)

¹Reprinted Marsh (ed.) Logic and Knowledge (George Allen & Unwin, 1956). All page references are to this work unless otherwise indicated.
2 The Vicious Circle Principle

Russell’s diagnosis of the paradoxes

Russell identifies a common element in the paradoxes: ‘self-reference’ or ‘reflexiveness’:

The remark of Epimenides must include itself in its own scope. If all classes, provided they are not members of themselves, are members of w, this must also apply to w. (p. 61)

(1) Russell’s diagnosis of the Liar

When a man says ‘I am lying’, we may interpret his statement as... ‘It is not true for all propositions p that if I affirm p, p is true’. The paradox results from regarding this statement as affirming a propositions which must therefore come within the scope of the statement. (pp. 61–2)

To elaborate, suppose:

(a) \( q = \neg \text{Tr}(\forall p(\text{I affirm}(\neg p) \rightarrow \text{Tr}(\neg p))) \)

(b) \( \exists ! p(\text{I affirm}(\neg p) \land p = q) \).

Russell: problem: \( q \) occurs in the range of \( \forall p \).

\( \text{Tr}(\neg q) \)

iff \( \forall p(\text{I affirm}(\neg p) \rightarrow \text{Tr}(\neg p)) \) (by (b))

iff \( \text{Tr}(\forall p(\text{I affirm}(\neg p) \rightarrow \text{Tr}(\neg p))) \) (by (\( \ast \))\))

iff \( \neg q \) (by (a))

iff \( \neg \text{Tr}(\neg q) \) (by (\( \ast \))\))

Note we rely on (\( \ast \)): \( \forall p(\text{Tr}(\neg p) \leftrightarrow p) \).

(2) Russell diagnosis of Russell’s paradox

the class \( w \) is defined by reference to ‘all classes’, and then turns out to be one among classes. (p. 62)

To elaborate: suppose:

(c) \( w = \{ x : x \notin x \} \)

(\( \ast \ast \)) \( \forall z(z \in \{ x : \phi(x) \} \leftrightarrow \phi(z)) \)

Then: \( w \in w \) iff (by (c)) \( w \in \{ x : x \notin x \} \) iff (by (\( \ast \ast \))) \( w \notin w \)
The Vicious Circle Principle (VCP)

Russell refines his account of the common element as follows:

all our contradictions have in common the assumption of a totality such that, if it were legitimate, it would at once be enlarged by new members defined in terms of itself. (p. 63)

Russell goes on to state the VCP, twice:

This leads us to the rule: \[ \text{VCP1} \] ‘Whatever involves all of a collection must not be one of the collection’; or, conversely: \[ \text{VCP2} \] ‘If, provided a certain collection had a total, it would have members only definable in terms of that total, then the said collections has no total’. (ibid.)

He adds in a footnote:

When I say that a collection has no total, I mean that statements about all its members are nonsense. (ibid.)

VCP applied: propositions

An application will make this clearer.

1 By \text{VCP1}: q presumably ‘involves’ the domain of \( \forall p \). Hence it cannot be in this domain.

2.a If \( \forall p \) meaningfully ranges over all propositions—‘if the collection of propositions has a total’—we can formulate \( q \) (and presumably \( q \) is ‘only definable’ this way).

2.b By \text{VCP2}: The collection of all propositions has no total: \( \forall p \) cannot meaningful range over all propositions.

2.c Instead \( \forall p \) can only range over a limited domain of propositions. (Later: a type.)

3 Simple types and Russell’s paradox

Russell’s positive proposal

Russell: VCP is purely negative; he seeks a positive resolution of the paradoxes (p. 63). His solution to Russell’s paradox has two components:

The theory of types propositions and propositional functions divide into different types. Quantifiers only ever range over a single type.

The no class theory of classes class-talk is elliptical for propositional-function-talk.
Propositional functions

Propositional functions are Russell’s analogue of Frege’s concepts.

A *propositional function* is simply any expression containing an undetermined constituent, or several undetermined constituents, and becoming a propositions as soon as the undetermined constituents are determined.’ (Lectures on logical atomism, p. 230)

Propositional functions contain free (‘real’) and bound (‘apparent’) variables. Propositions only contain bound variables.

**Examples**

‘If $x$ is man, then $x$ is mortal’  
prop. fn. real variables: $x$  
apparent variables: none

‘If Socrates is a man, Socrates is mortal’  
prop. real variables: none  
apparent variables: none

‘For all $x$: if $x$ is a man, $x$ is mortal’  
prop. real variables: none  
apparent variables: $x$

‘For all $y$: if $x$ is a man, $y$ is mortal’  
prop. fn. real variables: $x$  
apparent variables: $y$

**The simple theory (simplified)**

Focus first on propositional functions of a single argument.

Simple type theory: divides propositional functions into types according to the type of their free variables.

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<table>
<thead>
<tr>
<th>type</th>
<th>entity</th>
<th>example</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>individuals</td>
<td>Socrates, Russell</td>
</tr>
<tr>
<td>(0)</td>
<td>first level prop. fn</td>
<td>‘$x$ is Mortal’</td>
</tr>
<tr>
<td>((0))</td>
<td>second level prop. fn</td>
<td>$\exists x X(x), \forall Z \forall x (Zx \rightarrow Xx)$</td>
</tr>
</tbody>
</table>
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Syntax: an expression of the form $F(e)$ is meaningful only if $F$ has type $(\tau)$ and $e$ has type $\sigma$.

**The simple theory (more fully)**

In general: propositional functions may take multiple argument of different types.

e.g. ‘$x$ loves $y$’ has type $(0,0)$
e.g. $\exists x \exists y Rxy$ has type $((0,0))$

Syntax: an expression of the form $F(e_1, \ldots e_k)$ is meaningful only if $F$ is a propositional function of type $(\tau_1, \ldots, \tau_k)$, and $e_1, \ldots, e_k$ have respective types $\tau_1, \ldots, \tau_k$. 
The no class theory of classes

Russell: classes are ‘logical fictions’

Apparent reference to classes in the surface structure disappears at the level of logical form: \( y \in \{ x : \phi(x) \} =_d \phi(y) \)

(Special case: this applies only to extensional functions; Russell give a more complicated paraphrase in this spirit to account for intensional functions (p. 89).)

Example: Socrates \( \in \{ x : \text{Mortal}(x) \} \)

Surface form: the expression appears to be the result applying ‘\( \in \)’, type \((0,0)\), to two type 0 expressions: ‘Socrates’ and ‘\( \{ x : \text{Mortal}(x) \} \)’

Logical form: in fact, it’s true form is the result of applying ‘\( \text{Mortal}(x) \)’, type \((0)\), to one type 0 expression: ‘Socrates’, i.e. it has the form: \( \text{Mortal}(\text{Socrates}) \).

Application: Russell’s paradox.

This dissolves Russell’s paradox: classes are eliminated in favour of propositional functions, and Russell’s logic governing these is consistent.

consider the classes which are not members of themselves. It is plain that since we have identified classes with functions, no class can be significantly said to be or not to be a member of itself; for the members of a class are arguments to it, and arguments to a function are always of lower type than the function. (p. 88)

To elaborate: the closest we can come to \( x \notin x \) is the manifestly illformed \( \neg \phi(\phi) \).

To see the matter from another angle:

1. The naive axioms for sets (\( \star \star \)) \( \forall z (z \in \{ x : \phi(x) \} \leftrightarrow \phi(z)) \) is only inconsistent if we ascribe it the logical form it appears to have.

2. But on Russell’s account, it has a quite different logical form: \( \forall z (\phi(z) \leftrightarrow \phi(z)) \).

3. This clearly generates no contradiction.

\(^2\)See e.g. his Introduction to mathematical philosophy, ch. 17, and ‘Lecture on logical atomism’ in Marsh (ed) Logic and Knowledge, pp. 191, 265.
4 Russell’s Logicism

Russell: constructs the natural numbers in a similar way to Frege

\[
0 = \{X : X \approx \emptyset\} = \{\emptyset\} \\
1 = \{X : X \approx \{\text{Russell}\}\} = \{\{\text{Russell}\}, \{\text{Socrates}\}, \ldots\} \\
2 = \{X : X \approx \{\text{Russell, Socrates}\}\} = \{\{\text{Russell, Socrates}\}, \{\text{Russell, Frege}\}, \ldots\} \\
\vdots
\]

Immediately precedes \((P)\) can be defined much as before, and then Russell can define natural number as follows:

\[N x \equiv \forall F (F 0 \land \forall s \forall t (P s t \rightarrow (F s \rightarrow F t)) \rightarrow F x)\]

Important differences: Frege’s construction vs Russell’s

Frege: Natural numbers are objects: extensions of concepts.

Russell: Natural numbers are propositions functions: numbers are classes of classes, i.e. \(n + 1\)th level propositional functions (for \(n \geq 1\))

\[\text{e.g. Surface form: (i) } \text{Russell} \in \{\text{Russell}\} \text{ and } \{\text{Russell}\} \in 1 \]

\[\text{Logical form: (i) } F(\text{Russell}) \text{ and (ii) One}(F)\]

This difference has two important consequences:

The axiom of infinity

Russell: needs to assume the existence of an infinity of individuals to ensure the infinity of the natural numbers.

For suppose there were only \(n\) individuals altogether in the universe, where \(n\) is finite. There would then be \(2^n\) classes of individuals, and \(2^{2^n}\) classes of classes of individuals, and so on. Thus the cardinal number of terms in each type wold be finite; and though these numbers would grow beyond any assigned finite number, there would be no way of adding them so as to get an infinite number. Hence we need an axiom, so it would seem, to the effect that no finite class of individuals contains all individuals; but if any one chooses to assume that the total number of individuals in the universe is (say) 10,367, there seems no \textit{a priori} way of refuting his opinion. (p. 97.)

Cost: this seemingly vitiates Fregean epistemic motivations for logicism.

Russell: conditionalize theorems requiring the axiom.

\[\text{e.g. ‘There are infinitely many primes’ becomes ‘If there are infinitely many individuals, then there are infinitely many primes’.}\]
Typical ambiguity

Each number, identified with equivalence classes of similar classes, occurs at every level of the type-theoretic hierarchy (above 1).

2$_2$ = the class of all classes of pairs of individuals (a second-level class)
2$_3$ = the class of all classes of pairs of first-level classes (a third-level class)
2$_4$ = the class of all classes of pairs of second-level classes (a fourth-level class)

Bostock: does postulating infinitely many types of 2 have a consonance with common sense that makes Russell’s theory inherently credible?³

5 Ramified types and the Liar

Simple type theory: The Liar is still inconsistent

Unlike classes, propositions don’t disappear on analysis. The Liar sentence $q$ has the logical form it appear to

$$q = \neg Tr(\forall p (I\text{ affirm}('p') \rightarrow Tr('p')))$$

This leads to a contradiction in Simple Type Theory (given the earlier auxiliary assumptions, (b) and (⋆)).

Simple vs. Ramified type theory

Russell’s response: ramify the hierarchy.

Simple type theory

• propositions and propositional functions are divided into types according to the type of their free (‘real’) variables.
• Quantifiers only ever range over a single type.

Ramified type theory

• propositions and propositional functions are divided into types according to both the types of their free and bound (‘apparent’) variables.
• Quantifiers only ever range over a single order of a single type.

³A Study of Type-Neutrality’ Journal of Philosophical Logic 9 (1980).
The ramified hierarchy of propositions

To simplify, focus on a single simple type: propositions. The ramified hierarchy divides this type into infinitely many orders.

<table>
<thead>
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</thead>
<tbody>
<tr>
<td>0th</td>
<td>individuals</td>
<td>Socrates, Russell</td>
</tr>
<tr>
<td>1st</td>
<td>predicative propositions with bound variables (if any) ranging only over individuals</td>
<td>$\forall x(x \text{ is mortal})$</td>
</tr>
<tr>
<td>2nd</td>
<td>propositions containing bound variables ranging over 1st order (and no 2nd or higher order) propositions</td>
<td>$\exists p^1(\text{ I affirm } (p^1))$</td>
</tr>
<tr>
<td>3rd</td>
<td>propositions containing bound variables ranging over 2nd order (and no 3rd or higher order) propositions</td>
<td>$\exists p^2(\text{ I affirm } (p^2))$</td>
</tr>
</tbody>
</table>

Russell’s resolution of the Liar

‘Wherever ‘all propositions’ are mentioned, we must substitute ‘all propositions of order \(n\)’... Thus when a man says ‘I am lying’ we must interpret him as meaning: ‘There is a propositions of order \(n\), which I affirm, and which is false.’ This is a proposition of order \(n + 1\); hence the man is not affirming any proposition of order \(n\); hence his statement is false, and yet its falsehood does not imply, as that of ‘I am lying’ appear to do, that he is making a true statement. This solves the liar. (p. 79.)

To elaborate, consider:

\[ r^{n+1} = \exists p^n(\text{ I affirm } (p^n)) \land \neg Tr'(p^n) \]

Assuming $\neg \exists p^n(\text{ I affirm } (p^n))$; then this is unproblematically false.

(Similarly $q^{n+1} = \neg Tr'(\forall p^n(\text{ I affirm } (p^n)) \rightarrow Tr'(p^n))$ comes out unproblematically false, given (\(\star\))
6 The axiom of reducibility

Problem: the ramified type theory hampers mathematics

... any statement about ‘all properties of \( x \)’ is meaningless. ... But it is absolutely necessary, if mathematics is to be possible, that we should have some method of making statements which will usually be equivalent to what we have in mind when we (inaccurately) speak of ‘all properties of \( x \)’ (p. 80)

Example: definition of natural number

Russell: this causes trouble for the definition of ‘natural number’ (ibid.)

Suppose we define natural number for first-order type (0) functions (alias ‘predicative’ type(0) functions.):

\[ N^2 x = \forall F^1 0 \land \forall s \forall t (P^st \rightarrow (F^1 \rightarrow F^1 t)) \rightarrow F^1 x \]

Russell: we cannot then apply induction to second-order properties (e.g. \( N^2 \)).

Myhill: there is no way to define ‘natural number’ in ramified type theory.\(^4\)

Russell: solves this problem by adding a further axiom

Axiom of reducibility ‘every function is equivalent, for all its values, to some predicative function of the same argument.’ (p. 81)

Motivating reducibility: classes

[Reducibility] seems to be what common sense effects by the admission of classes. Given any propositional function \( \phi x \), of whatever order, this is assumed to be equivalent, for all values of \( x \), to a statement of the form ‘\( x \) belongs to the class \( \alpha \)’. Now this statement is of the first order, since it makes no allusion to ‘all functions of such-and-such a type’. (p. 81.)

Criticisms of Reducibility

Weyl: [Reducibility] ‘is a bold, an almost fantastic axiom; there is little justification for it in the real world in which we live’\(^5\)

Ramsey, Putnam: reducibility undoes the effect of the ramifying the hierarchy.\(^6\)

Russell: the paradoxes do not reemerge.

E.g. even if \( q^2 \) is equivalent to \( q^1 \), asserting \( q^2 \) is not equivalent to asserting \( q^1 \).

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\(^4\)The Undefinability of the Set of Natural Numbers in the Ramified Principia’ in G. Nakhnikian (ed.) Bertrand Russell’s Philosophy (Duckworth, 1974).
